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TRANSIENT ROTARY SHEAR WAVES IN NONHOMOGENEOUS VISCOELASTIC MEDIA

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Abstract—Propagation of axisymmetrical rotary shear waves in nonhomogeneous viscoelastic media with a cylindrical hole is studied. Nonhomogeneities are assumed to depend on the radial distance from the axis of the hole. By means of the theory of the propagating surfaces of discontinuities, the solutions for the shear stress and particle velocity are expanded as Taylor series about the time of arrival of the wave front. Both stress-prescribed and velocity-prescribed boundary conditions are considered. The corresponding elastic problem is investigated as a special case; and, for certain types of nonhomogeneity, explicit solutions are presented and compared with the solutions obtained by the Laplace transform technique.

INTRODUCTION

THE transient wave propagation in nonhomogeneous elastic media has drawn much attention recently. Sternberg and Chakravorty [1] investigated the propagation of rotary shear waves in a nonhomogeneous isotropic elastic plate of infinite extent with a circular opening. In [1], it was assumed that the shear modulus of the material was proportional to an arbitrary power of the radial distance from the center of the hole. Solutions were obtained by the Laplace transform technique. Chou and Schaller [2] later reconsidered the problem by employing the method of characteristics to develop a numerical scheme which was feasible for a wider class of nonhomogeneities. An analogous problem of axial shear wave propagation in nonhomogeneous elastic media has been investigated by Chou and Gordon [3] and later by Reddy and Marietta [4] by employing different approaches. Several authors [5–8] obtained solutions for longitudinal waves in various nonhomogeneous elastic rods of finite and semi-finite lengths.

A survey of the literature indicates that while propagation of transient waves in nonhomogeneous elastic media has received considerable attention, few problems of transient wave propagation in nonhomogeneous viscoelastic media have been investigated. Only recently Reiss [9] has studied the propagation of one-dimensional stress discontinuities in nonhomogeneous viscoelastic media by employing theory of weak solution. Since he used the model representation for the viscoelastic constitutive relation, the analysis was limited to less realistic materials.

In the present paper, the propagation of transient cylindrical shear waves in nonhomogeneous viscoelastic bodies is considered. The constitutive relation of the material is described by a creep function in shear. The theory of propagating surfaces of discontinuity is employed. Both the solutions for the stress and the particle velocity are expressed as Taylor's expansions about the wave front. Two types of boundary conditions with stress and particle velocity prescribed are considered. This analysis leads us to study the stress discontinuities of various order. The method, as is employed here, was applied by Achenbach and Reddy [10] to solve a wave propagation problem in a homogeneous viscoelastic rod, and, later, by Reddy [8] and Reddy and Marietta [4] to investigate the longitudinal wave propagation in a nonhomogeneous semi-infinite elastic rod and the axial shear wave propagation in a nonhomogeneous elastic medium.

The elastic problem considered by Sternberg and Chakravorty [1] naturally becomes a special case of the present investigation. The type of nonhomogeneity and the boundary conditions, as treated in this paper, are, however, more general than that of [1]. As examples, elastic materials with simple power distribution of the shear modulus under a step stress pulse are considered. For certain cases, where closed form solutions are obtainable by the Laplace transform technique, it is shown that the same solutions can be obtained by the present method. A numerical comparison between the present analysis and [1] is done for a particular case for which there exists no closed form solution. It is found that a few terms in the Taylor expansion suffice to give an accurate solution for the stress that is sufficiently near the wave front.

FORMULATION OF THE PROBLEM

For an axisymmetrical displacement field of pure rotary shear, the nontrivial displacement component and the equation of motion in cylindrical coordinates (r, θ, z) are

$$u_\theta = u(r, t) \tag{1}$$

and

$$\frac{\partial \tau}{\partial r} + \frac{2\tau}{r} = \rho \frac{\partial^2 u}{\partial t^2} \tag{2}$$

respectively. In equation (2), τ is the nonvanishing shear stress and $\rho(r)$ is the mass density. The physical problem governed by equations (1) and (2) may be interpreted as a plate of infinite extent with a circular hole or an infinite medium with a cylindrical port subjected to an axisymmetrical rotary disturbance. For convenience, we choose, in the sequel, the former case for discussion.

Consider a nonhomogeneous, linearly viscoelastic plate with a circular hole of radius a. We assume that the plate is in a state of plane stress. The material properties are assumed to depend solely on the radial distance from the center of the hole. If the medium is at rest prior to $t = 0$, the stress–strain relation can be expressed in the form

$$2\varepsilon_{r\theta}(r, t) = J_0(r)\tau(r, t) + \int_{0+}^{t} J^{(1)}(r, t-s)\tau(r, s)\,\mathrm{d}s \tag{3}$$

where the shearing strain is defined as

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right) \tag{4}$$

and $J(r, t)$ is the creep function in shear. In equation (3) we also define

$$J^{(n)}(r, t) = \frac{\partial^n J(r, t)}{\partial t^n}, \quad J_0 = J(r, 0), \quad J_0^{(n)} = \left.\frac{\partial^n J(r, t)}{\partial t^n}\right|_{t=0}. \tag{5}$$

We first introduce the following new variables:

$$\sigma = r^2\tau \tag{6}$$

$$m = r^3\rho \tag{7}$$

$$U = u/r, \qquad V = v/r \tag{8}$$

$$B^{(n)}(r, t) = J^{(n)}(r, t)/r^3 \qquad B_0 = J_0/r^3, \qquad B_0^{(n)} = J_0^{(n)}/r^3. \tag{9}$$

In equation (8), v is the tangential particle velocity. By virtue of the above modified quantities, we are allowed to rewrite the governing equations, equation (2) and (3), in simpler forms as

$$\frac{\partial \sigma}{\partial r} = m\frac{\partial^2 U}{\partial t^2} \tag{10}$$

$$\frac{\partial U}{\partial r} = B_0(r)\sigma(r, t) + \int_{0+}^{t} B^{(1)}(r, t-s)\sigma(r, s)\,\mathrm{d}s \tag{11}$$

respectively. Thus, we need to determine the solutions for equations (10) and (11) which are conformable to the appropriate boundary conditions. In this paper, two types of boundary conditions will be considered:

(a) The shearing stress at the hole, $\tau(a, t)$, is prescribed and can be represented by a Maclaurin expansion

$$\tau(a, t) = \sum_{n=0}^{\infty} \tau_n \frac{t^n}{n!} \qquad t > 0. \tag{12}$$

(b) The tangential particle velocity at $r = a$, $v(a, t)$, is prescribed and can be expanded as

$$v(a, t) = \sum_{n=0}^{\infty} v_n \frac{t^n}{t!} \qquad t > 0. \tag{13}$$

PROPAGATION OF THE WAVE FRONT

The wave front is defined as the surface which travels through the medium as t varies continuously, and across which there may exist a discontinuity in the stress, particle velocity and their time derivatives. In view of the axisymmetry of the problem, we may suppose that the position of the wave front is given by

$$r = \psi(t). \tag{14}$$

Then the velocity of the wave front relative to the material is obtained as

$$c = \frac{\mathrm{d}\psi}{\mathrm{d}t}. \tag{15}$$

From equation (15), we obtain

$$t = \phi(r) = \int_a^r \frac{\mathrm{d}r}{c}. \tag{16}$$

Basic to the study of propagating discontinuities is the kinematical condition of compatibility which is discussed in general by Thomas [11]. For a function $f(r, t)$ which is discontinuous and has discontinuous derivatives across the wave front that moves in the radial direction with velocity c, the kinematic condition of compatibility takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}[f] = \left[\frac{\partial f}{\partial t}\right] + c\left[\frac{\partial f}{\partial r}\right] \tag{17}$$

where, in the usual manner, finite jumps across the wave front are denoted by square brackets.

In this paper, it is assumed that the integrity of the material is not affected by the propagation of the stress, i.e. the displacement remains continuous. Thus

$$[u] = 0 \tag{18}$$

across the wave front. From equation (8), it is noted that the modified displacement U is also continuous across the wave front. With the foregoing in mind, we apply the general rule given by equation (17) to U to obtain

$$\left[\frac{\partial U}{\partial r}\right] = -\frac{1}{c}\left[\frac{\partial U}{\partial t}\right]. \tag{19}$$

Since the integral in equation (11) is continuous at the wave front, we have the relation

$$\left[\frac{\partial U}{\partial r}\right] = B_0[\sigma]. \tag{20}$$

Elimination of $[\partial U/\partial r]$ from equations (19) and (20) yields

$$[\sigma] = -\frac{1}{B_0 c}\left[\frac{\partial U}{\partial t}\right]. \tag{21}$$

Conservation of linear momentum at the wave front was discussed by Thomas [11] in a general form. For the present problem, it can be expressed in the form

$$[\tau] = -\rho c[v]. \tag{22}$$

In terms of the modified stress σ, the modified mass density m and the modified particle velocity V, equation (22) becomes

$$[\sigma] = -mc[V]. \tag{23}$$

Comparing equation (21) with (23), we obtain

$$c^2 = \frac{1}{B_0 m} = \frac{1}{J_0(r)\rho(r)}. \tag{24}$$

Thus, the wave front propagates with a velocity which depends on the glassy compliance and the mass density, and, consequently, may vary as it penetrates into the medium.

PROPAGATION OF ROTARY SHEAR WAVES

The transient axisymmetrical shear wave propagation problem is governed by equations (10) and (11) together with the quiescent initial conditions and the boundary conditions as given by equations (12) and (13). Following Achenbach and Reddy [10] we now seek the solutions for the stress and the particle velocity as Taylor's expansions about the time of arrival of the wave front. In terms of the modified field quantities, σ and V, we write

$$\sigma(r,t) = \sum_{n=0}^{\infty} \frac{1}{n!}\{t-\phi(r)\}^n \left[\frac{\partial^n \sigma}{\partial t^n}\right]_{t=\phi(r)} \qquad t \geq \phi(r) \tag{25}$$

$$V(r,t) = \sum_{n=0}^{\infty} \frac{1}{n!}\{t-\phi(r)\}^n \left[\frac{\partial^n V}{\partial t^n}\right]_{t=\phi(r)} \qquad t \geq \phi(r). \tag{26}$$

The problem is then reduced to seeking solutions for the propagating discontinuities, or the coefficients of expansion in equations (25) and (26).

We first differentiate equation (11) with respect to t for $n+1$ times to obtain

$$\frac{\partial^{n+2} U}{\partial r\, \partial t^{n+1}} = B_0(r)\frac{\partial^{n+1}\sigma}{\partial t^{n+1}} + \sum_{i=1}^{n+1} B_0^{(i)} \frac{\partial^{n+1-i}\sigma}{\partial t^{n+1-i}} + \int_{0^+}^{t} B^{(n+1)}(r, t-s)\sigma(r,s)\,\mathrm{d}s. \tag{27}$$

Since the integral is continuous at the wave front, equation (27) yields the following relation between the discontinuities

$$\left[\frac{\partial^{n+2} U}{\partial r\, \partial t^{n+1}}\right] = B_0\left[\frac{\partial^{n+1}\sigma}{\partial t^{n+1}}\right] + \sum_{i=1}^{n+1} B_0^{(i)}\left[\frac{\partial^{n+1-i}\sigma}{\partial t^{n+1-i}}\right]. \tag{28}$$

Applying the kinematic condition of compatibility as given by equation (17) to the derivatives $\partial^{n+1}U/\partial^{n+1} (n \geq 0)$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial^{n+1} U}{\partial t^{n+1}}\right] = \left[\frac{\partial^{n+2} U}{\partial t^{n+2}}\right] - c\left[\frac{\partial^{n+2} U}{\partial r\, \partial t^{n+1}}\right]. \tag{29}$$

Elimination of $[\partial^{n+2}U/\partial r \partial t^{n+1}]$ from equations (28) and (29) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial^{n+1} U}{\partial t^{n+1}}\right] - \left[\frac{\partial^{n+2} U}{\partial t^{n+2}}\right] = cB_0\left[\frac{\partial^{n+1}\sigma}{\partial t^{n+1}}\right] + \sum_{i=1}^{n+1} cB_0^{(i)}\left[\frac{\partial^{n+1-i}\sigma}{\partial t^{n+1-i}}\right]. \tag{30}$$

For $n = 0$, equation (30) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial U}{\partial t}\right] - \left[\frac{\partial^2 U}{\partial t^2}\right] = cB_0\left[\frac{\partial \sigma}{\partial t}\right] + B_0^{(1)}[\sigma]. \tag{31}$$

From equation (10) we have

$$\left[\frac{\partial \sigma}{\partial r}\right] = m\left[\frac{\partial^2 U}{\partial t^2}\right]. \tag{32}$$

By employing the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}[\sigma] = \left[\frac{\partial \sigma}{\partial t}\right] + c\left[\frac{\partial \sigma}{\partial r}\right] \tag{33}$$

equation (32) can be written as

$$\left[\frac{\partial^2 U}{\partial t^2}\right] = \frac{1}{mc}\left\{\frac{\mathrm{d}}{\mathrm{d}t}[\sigma] - \left[\frac{\partial \sigma}{\partial t}\right]\right\}. \tag{34}$$

We now substitute equations (23) and (34) in equation (31) to obtain, after some manipulation,

$$\frac{\mathrm{d}}{\mathrm{d}t}[\sigma]+\alpha(t)[\sigma] = 0, \tag{35}$$

where

$$\alpha(t) = \frac{1}{2}\frac{J_0^{(1)}}{J_0}-\frac{1}{2mc}\frac{\mathrm{d}}{\mathrm{d}t}(mc). \tag{36}$$

For $n \geq 1$, the discontinuities of the time derivatives of U on the left-hand side of equation (30) can be replaced by the derivatives of σ as follows. We differentiate equation (10) with respect to time for $n-1$ times to obtain

$$\left[\frac{\partial^n \sigma}{\partial r\,\partial t^{n-1}}\right] = m\left[\frac{\partial^{n+1} U}{\partial t^{n+1}}\right]. \tag{37}$$

Using relation (37) we are able to eliminate the time derivatives of U from equation (30). We obtain, for $n \geq 1$,

$$\frac{1}{c}\left[\frac{\partial^n \sigma}{\partial r\,\partial t^{n-1}}\right]\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{m}\right)+\frac{1}{mc}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial^n \sigma}{\partial r\,\partial t^{n-1}}\right]-\frac{1}{mc}\left[\frac{\partial^{n+1}\sigma}{\partial r\,\partial t^n}\right] = B_0\left[\frac{\partial^{n+1}\sigma}{\partial t^{n+1}}\right]$$
$$+\sum_{i=1}^{n+1} B_0^{(i)}\left[\frac{\partial^{n+1-i}\sigma}{\partial t^{n+1-i}}\right]. \tag{38}$$

By writing the kinematic condition of compatibility for $\partial^{n-1}\sigma/\partial t^{n-1}$ and $\partial^n\sigma/\partial t^n$, and employing the relation

$$B_0 = \frac{1}{mc^2} \tag{39}$$

we obtain from equation (38) the following linear ordinary differential equation for $[\partial^n\sigma/\partial t^n]$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial^n\sigma}{\partial t^n}\right]+\alpha(t)\left[\frac{\partial^n\sigma}{\partial t^n}\right] = F_n(t) \quad \text{for} \quad n \geq 1. \tag{40}$$

In equation (40), $\alpha(t)$ is given by equation (36), and $F_n(t)$ is defined as

$$F_n(t) = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\frac{\partial^{n-1}\sigma}{\partial t^{n-1}}\right]-\frac{1}{2}\left\{\frac{1}{mc}\frac{\mathrm{d}}{\mathrm{d}t}(mc)\right\}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial^{n-1}\sigma}{\partial t^{n-1}}\right]-\frac{1}{2}\sum_{i=2}^{n+1}\frac{B_0^{(i)}}{B_0}\left[\frac{\partial^{n+1-i}\sigma}{\partial t^{n+1-i}}\right]. \tag{41}$$

If we define $F_n(t) = 0$ for $n = 0$, the general solutions for equations (35) and (40) can be written in a single expression as

$$\left[\frac{\partial^n\sigma}{\partial t^n}\right] = \mathrm{e}^{-\beta(t)}\left\{\int_0^t F_n(s)\mathrm{e}^{\beta(s)}\,\mathrm{d}s+A_n\right\} \; n \geq 0 \tag{42}$$

where

$$\beta(t) = \int_0^t \alpha(s)\,\mathrm{d}s, \tag{43}$$

and $A_n (n \geq 0)$ are integration constants to be determined by the boundary conditions. The growth or decay of the magnitudes of propagating discontinuities is determined by the sign of $\beta(t)$. In view of equation (36), it is evident that all such disturbances in a homogeneous viscoelastic medium always attenuate, since for linear viscoelastic solids experiments have confirmed the positive nature of J_0 and $J_0^{(1)}$, the glassy compliance and the initial slope of the creep function. We also observe from equation (36) that it is possible for $\beta(t)$ to assume negative values for certain types of nonhomogeneity. Thus, the shear wave may also grow as it propagates into a nonhomogeneous viscoelastic medium.

A close examination of equations (41) and (42) reveals that the solution of $[\partial^n \sigma/\partial t^n]$ depends on the solution of $[\partial^{n-1}\sigma/\partial t^{n-1}]$. For $n = 0$, the solution is

$$[\sigma] = A_0 \, \mathrm{e}^{-\beta(t)}. \tag{44}$$

For $n \geq 1$, the solutions are obtained from equation (42) together with equations (41) and (44).

Before we proceed to determine the arbitrary constants, A_n, it is necessary to determine the coefficients, $[\partial^n V/\partial t^n]$, of the Taylor's expansion for the modified particle velocity V as given by equation (26). Since the solutions for $[\partial^n \sigma/\partial t^n]$ $(n \geq 0)$ have been obtained, we will express $[\partial^n V/\partial t^n]$ in terms of $[\partial^n \sigma/\partial t^n]$ and their time derivatives. From equation (23), we have

$$[V] = -\frac{1}{mc}[\sigma]. \tag{45}$$

For $n \geq 1$, we employ the rule as given by equation (17) to $\partial^{n-1}\sigma/\partial t^{n-1}$. We write

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial^{n-1}\sigma}{\partial t^{n-1}}\right] = \left[\frac{\partial^n \sigma}{\partial t^n}\right] + c\left[\frac{\partial^n \sigma}{\partial r\, \partial t^{n-1}}\right]. \tag{46}$$

Substitution of equation (37) in equation (46) yields

$$\left[\frac{\partial^n V}{\partial t^n}\right] = \frac{1}{mc}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial^{n-1}\sigma}{\partial t^{n-1}}\right] - \left[\frac{\partial^n \sigma}{\partial t^n}\right]\right\}. \tag{47}$$

In deriving equation (47), the relation $V = \partial U/\partial t$ was used. In view of equation (47), the Taylor's expansion for V can now be written as

$$V(r, t) = \sum_{n=0}^{\infty} -\frac{1}{n!}\{t - \phi(r)\}^n \left\{\frac{1}{mc}\left[\frac{\partial^n \sigma}{\partial t^n}\right] - \frac{1}{mc}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial^{n-1}\sigma}{\partial t^{n-1}}\right]\right\}_{t=\phi(r)} \tag{48}$$

in which we set

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial^{n-1}\sigma}{\partial t^{n-1}}\right] \equiv 0 \text{ if } n = 0. \tag{49}$$

This completes the general solutions for the stress and the particle velocity.

The two types of boundary condition as described by equations (12) and (13) are now employed to determine the constants of integration in equation (42). For type A of the boundary conditions, the stress is prescribed at $r = a$, and we can write

$$\sigma(a, t) = \sum_{n=0}^{\infty} \sigma_n \frac{t^n}{n!} \qquad t > 0 \tag{50}$$

in which the coefficients σ_n are given by

$$\sigma_n = a^2\tau_n. \tag{51}$$

Comparing equation (50) with equation (25) at $r = a$ in conjunction with equation (42) we obtain

$$A_n = \sigma_n. \tag{52}$$

For type B of the boundary conditions, the particle velocity at the boundary surface $(r = a)$ is expanded in a Maclaurin series as given by equation (13). Equation (13) may also be written in terms of the modified particle velocity V, i.e.

$$V(a, t) = \sum_{n=0}^{\infty} V_n \frac{t^n}{n!} \qquad t > 0 \tag{53}$$

where V_n are defined as

$$V_n = v_n/a. \tag{54}$$

A comparison between equation (53) and equation (48) at $r = a$ yields

$$\left[\frac{\partial^n \sigma}{\partial t^n}\right]_{t=0} - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial^{n-1}\sigma}{\partial t^{n-1}}\right]_{t=0} = -mcV_n. \tag{55}$$

It is noted that in equations (55) the relation $t = \phi(r)$ at the wave front is employed. From equation (42), we have

$$A_n = \left[\frac{\partial^n \sigma}{\partial t^n}\right]_{t=0}. \tag{56}$$

Substitution of equation (56) in equation (55) yields

$$A_n = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial^{n-1}\sigma}{\partial t^{n-1}}\right]_{t=0} - mcV_n, \qquad n \geq 0. \tag{57}$$

In view of equation (49), we obtain the value of the first constant

$$A_0 = -mcV_0. \tag{58}$$

It is observed that the solution for $[\partial^{n-1}\sigma/\partial t^{n-1}]$ as given by equation (42) involves only the constants $A_i (i \leq n-1)$. Thus, all constants A_n can be determined successively according to equation (57) if the first constant A_0 is known.

NONHOMOGENEOUS ELASTIC MEDIA

The governing equations as well as the solutions for waves in the nonhomogeneous elastic medium can be derived from those as have been described in the previous sections by requiring that

$$J^{(n)}(r, t) = 0 \text{ for } n \geq 1. \tag{59}$$

More specifically, the stress–strain relation for the corresponding elastic problem is

$$\frac{\partial u}{\partial r} - \frac{u}{r} = \frac{1}{\mu}\tau \tag{60}$$

where $\mu(r)$ is the shear modulus, and is a function of r only. It is easy to show that

$$J_0 = \frac{1}{\mu} \tag{61}$$

and that equation (11) becomes

$$\frac{\partial U}{\partial r} = B_0(r)\sigma(r, t) \tag{62}$$

in which

$$B_0(r) = 1/r^3\mu(r). \tag{63}$$

By substituting equation (59) in equation (36), we can carry out the integration of equation (43) to obtain

$$\beta(t) = -\frac{1}{2}\log\left(\frac{mc}{m_0c_0}\right) \tag{64}$$

where

$$m_0 = m|_{r=a} \qquad c_0 = c|_{r=a}. \tag{65}$$

It follows that equation (42) now takes the form

$$\left[\frac{\partial^n \sigma}{\partial t^n}\right] = \left(\frac{mc}{m_0c_0}\right)^{\frac{1}{2}}\left\{\int_0^t F_n(s)\left(\frac{mc}{m_0c_0}\right)^{-\frac{1}{2}} ds + A_n\right\} \tag{66}$$

in which

$$F_n(t) = \frac{1}{2}\frac{d^2}{dt^2}\left[\frac{\partial^{n-1}\sigma}{\partial t^{n-1}}\right] - \frac{1}{2}\left\{\frac{1}{mc}\frac{d}{dt}(mc)\right\}\frac{d}{dt}\left[\frac{\partial^{n-1}\sigma}{\partial t^{n-1}}\right]. \tag{67}$$

The problem of axisymmetrical shear wave propagation due to a suddenly applied uniform shearing traction in a nonhomogeneous elastic plate of infinite extent was studied by Sternberg and Chakravorty [1]. Solutions were obtained by the Laplace transform technique. Chou and Schaller [2] reconsidered the problem by employing the method of characteristics to develop a scheme for numerical solutions. In the sequel, we will consider the problem of [1] as a special case of the present analysis, and compare the present solutions with the solutions obtained in [1]. Following [1] the shear modulus of the nonhomogeneous plate is taken proportional to an arbitrary power of the radial distance from the axis of the opening, i.e.

$$\mu(r) = \mu_0\left(\frac{r}{a}\right)^\eta \tag{68}$$

where η is an arbitrary real number, and μ_0 is the value of μ at the hole. In order to shorten the analysis, the mass density ρ is regarded as constant. Using the expression for the shear modulus as given by equation (68), the following relations can be easily obtained:

$$B_0(r) = a^\eta/\mu_0\gamma^{3+\eta} \tag{69}$$

$$m = \rho r^3 \tag{70}$$

$$c = c_0(r/a)^{\eta/2} \tag{71}$$

$$c_0 = (\mu_0/\rho)^{\frac{1}{2}}. \tag{72}$$

If a stress of magnitude τ_0 is suddenly applied at $r = a$ and maintained constant thereafter, the boundary condition is then given by equation (50) with

$$\sigma_0 = a^2\tau_0, \qquad \sigma_n = 0 \quad \text{for } n \geq 1. \tag{73}$$

The position of the wave front satisfies

$$t = \phi(r) = \frac{a^{\eta/2}}{c_0}\int_a^r \frac{\mathrm{d}r}{r^{\eta/2}}. \tag{74}$$

The integration of equation (74) depends on whether $\eta \neq 2$ or $\eta = 2$. This leads us to deal separately with these two cases.

Case 1. $\eta \neq 2$

For $\eta \neq 2$, equation (74) yields

$$t = \phi(r) = \frac{2a^{\eta/2}}{(2-\eta)c_0}(r^{1-\eta/2} - a^{1-\eta/2}). \tag{75}$$

Solving equation (75) for r we obtain

$$r = a\left(\frac{2-\eta}{2a}c_0 t + 1\right)^{2/(2-\eta)}. \tag{76}$$

It is noted that equations (75) and (76) hold only at the wave front. With equation (76), the expressions of m and c at the wave front as given by equations (70) and (71), respectively, can be written as functions of time, and, subsequently, the total differentiation of the product mc with respect to time can be obtained. Equation (67) then assumes the form

$$F_n(t) = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\frac{\partial^{n-1}\sigma}{\partial t^{n-1}}\right] - \frac{6+\eta}{4a}c_0\left(\frac{2-\eta}{2a}c_0 t + 1\right)^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial^{n-1}\sigma}{\partial t^{n-1}}\right]. \tag{77}$$

Solutions for the discontinuities are obtained from equation (66) by straightforward integrations. The coefficients of expansion in equations (25) and (48) are subsequently obtained from the discontinuities by replacing t by $\phi(r)$. The general forms of the first two coefficients are

$$[\sigma]_{t=\phi(r)} = A_0\left(\frac{r}{a}\right)^{(6+\eta)/4} \tag{78}$$

$$\left[\frac{\partial\sigma}{\partial t}\right]_{t=\phi(r)} = \left\{A_1 - A_0 k\left(\frac{r}{a}\right)^{-(2-\eta)/2} + A_0 k\right\}\left(\frac{r}{a}\right)^{(6+\eta)/4} \tag{79}$$

In equation (79),

$$k = \frac{c_0(6+\eta)(-10+\eta)}{16a(2-\eta)}. \tag{80}$$

In view of equation (52) and the boundary condition, equation (73), we have

$$A_0 = \sigma_0, A_n = 0 \text{ for } n \geq 1. \tag{81}$$

For $\eta = -\frac{2}{3}$, the complete set of coefficients can be obtained as

$$[\sigma]_{t=\phi(r)} = \left(\frac{r}{a}\right)^{4/3} \sigma_0 \tag{82}$$

$$\left[\frac{\partial^n \sigma}{\partial t^n}\right]_{t=\phi(r)} = (-1)^n \left(\frac{4c_0}{3a}\right)^n \left\{\left(\frac{r}{a}\right)^{4/3} - 1\right\}\sigma_0 \qquad n \geq 1. \tag{83}$$

The solution for the modified stress σ is

$$\frac{\sigma}{\sigma_0} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(\xi^{4/3}-1)\left\{\frac{4}{3}(t^*-\delta)\right\}^n - 1 = (\xi^{4/3}-1)\exp\left\{-\frac{4}{3}(t^*-\delta)\right\} - 1 \qquad t^* \geq \delta \tag{84}$$

where

$$\xi = r/a \tag{85}$$

$$t^* = c_0 t/a \tag{86}$$

$$\delta = \frac{3}{4}(\xi^{4/3}-1). \tag{87}$$

By using relations (6) and (51), we write the solution for the shear stress τ in the dimensionless form

$$\tau^* = (\xi^{-2/3} - \xi^{-2})\exp\{-\tfrac{4}{3}(t^*-\delta)\} - \xi^{-2} \tag{88}$$

where

$$\tau^* = \tau/\tau_0 \tag{89}$$

is the dimensionless shear stress. This solution was obtained earlier by Sternberg and Chakravorty [1], who used the Laplace transform technique.

The solution for the particle velocity may be similarly computed. It can be shown that the closed form solution for v according to the present analysis also agrees with the solution of [1].

For the case $\eta = 10$, we obtain

$$\left[\frac{\partial^n \sigma}{\partial t^n}\right]_{t=\phi(r)} = A_n \left(\frac{r}{a}\right)^4 \qquad n \geq 0. \tag{90}$$

Since the boundary condition requires that $A_0 = \sigma_0$ and $A_i = 0$ for $i \geq 1$, it is evident that

$$[\sigma] = \sigma_0 \xi^4 \tag{91}$$

is the only non-vanishing coefficient of expansion. The solution for the shear stress is then

$$\tau^* = \xi^2 \qquad t^* \geq \tfrac{1}{4}(1-\xi^{-4}) \tag{92}$$

which agrees with the solution obtained in [1]. It is noted that the stress grows as it propagates into the medium, though there is geometrical damping.

Case 2. $\eta = 2$

For this case, the time of arrival of the wave front at r is given by

$$t = \phi(r) = \frac{a}{c_0} \log \frac{r}{a}. \tag{93}$$

Inverting equation (93), we obtain

$$r = a\, e^{c_0 t/a}. \tag{94}$$

With equations (93) and (94) obtained, it is straightforward to calculate the coefficients of expansion. The dimensionless solution for the shear stress is obtained as

$$\tau^* = \sum_{n=0}^{\infty} \frac{1}{n!}(t^* - \gamma)^n b_n \qquad t^* > \gamma \tag{95}$$

where

$$\gamma = \log \xi \tag{96}$$

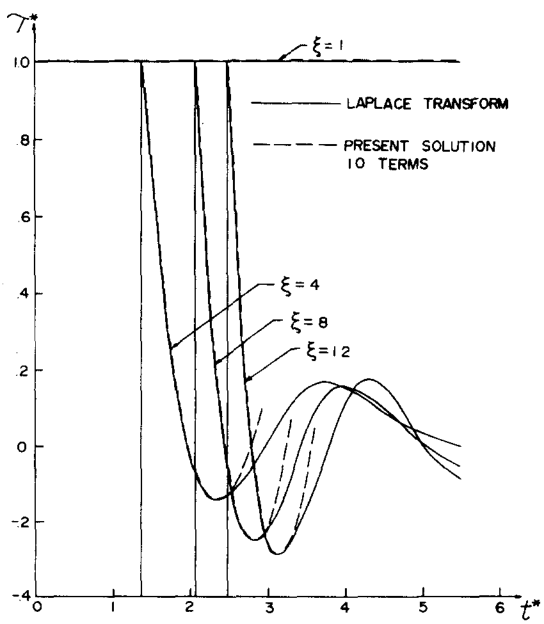

FIG. 1. Stress as a function of time at various positions for $\eta = 2$.

and the first five coefficients of b_n are

$$b_0 = 1$$
$$b_1 = -2\gamma$$
$$b_2 = 2\gamma^2$$
$$b_3 = 2\left(-\frac{4}{3}\gamma^3+\gamma\right)$$
$$b_4 = 4\left(\frac{4}{4!}\gamma^4-\gamma^2\right).$$

In Fig. 1, the dimensionless stress τ^* is plotted against t^* for several values of ξ. The first ten terms in the series are retained. For comparison, the solution obtained by Sternberg and Chakravorty [1] is also shown in the same figure. It is found that the present 10-term solution compares favorably with the solution of [1] over the time range that is close to the time of arrival of the wave front. This range can obviously be widened if more terms in the series are taken.

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Абстракт—Исследуется распространение осеимметрических ьраиуамебыых волн сдвига в неоднородных вязкоупругих средах с цилиндрическим отверстием. Предполагается, что неоднородность зависит от радиального расстояния от оси отверстия. С помощью теории распространения поверхностей разрыва, расширяются решения для напряжений сдвига и скорости частиц, ькаяесмье рядоь Тейлора по времени прибытия фронта волны. Рассматриваются краевые условия, так для заданных напряжений, как и заданной скорости. В качестве специального случая, исследуется соответствующая упругая задача. Даются решения в явном виде для некоторых типов неоднородности асравниваются с решениями, полученными на основе преобразования Лапласа.